B.Sc III.

Page) Date - 28-04-2020

Topic → 1. Implicit Function

v. Imp

2. Implicit Function theorem for a real valued function of two real variables

Implicit Function →

Definition :→ Let $f(x,y)$ be a real function of two real variables x, y and let $y = g(x)$ be a function of x such that for every value of x for which $g(x)$ is defined,

$f(x, g(x)) = 0$ i.e $y = g(x)$ is a root of the functional equation $f(x,y) = 0$. then $y = g(x)$ is called an 'implicit function' defined by the functional equation $f(x,y) = 0$

Remarks :→

[The difficulty of actual determination of an analytical expression does not rule out the possibility of the existence of the implicit function defined by a functional equation even though actual determination may not be possible.]

* First differential co-efficient of an implicit function —

Let $u = f(x,y) = 0$

$$\Rightarrow du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\Rightarrow \frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \quad \text{since } f(x,y) = 0 = u$$

$$\Rightarrow \frac{\partial f}{\partial y}\frac{dy}{dx} = -\frac{\partial f}{\partial x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

this is a very useful formula for finding out $\frac{dy}{dx}$ when the relationship between x and y is given in the form of an implicit function.

Second differential co-efficient of an implicit function —

Usual notation —

$p = f_x$, $q = f_y$

$r = f_{xx}$, $s = f_{yx}$

$t = f_{yy}$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-f_x}{f_y} = -\frac{p}{q}$$

$$\frac{d^2y}{dx^2} = -\frac{q^2r - 2pqs + p^2t}{q^3}$$

$$q^3 \frac{d^2y}{dx^2} = \begin{vmatrix} r & s & p \\ s & t & q \\ p & q & 0 \end{vmatrix}$$ Important

Ex:→ If. $ax^2 + 2hxy + by^2 = 1$ then find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Here $ax^2 + 2hxy + by^2 = 1$

$\Rightarrow f(x,y) = ax^2 + 2hxy + by^2 - 1 = 0$

$f_x = 2ax + 2hy = 2(ax + hy)$ $f_{xx} = 2a$

$f_y = 2hx + 2by = 2(hx + by)$ $f_{yy} = 2b$

$f_{yx} = 2h$ $t = f_{yy} = 2b$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\cancel{2}(ax+hy)}{\cancel{2}(hx+by)} = -\frac{ax+hy}{hx+by}$$

$$\frac{d^2y}{dx^2} = -\frac{q\,\frac{qr - ps}{q} - p\,\frac{qs - pt}{q}}{q^2} = -\frac{q^2r - 2pqs + p^2t}{q^3}$$

$$= -\frac{4(hx+by)^2 \cdot 2a - 2 \cdot 2(ax+hy) \cdot 2(hx+by) \cdot 2h + 4(ax+hy)^2 \cdot 2b}{4(hx+by)^3}$$

$$\therefore \frac{d^2y}{dx^2} = -2\left[\frac{(hx+by)^2 a - 2(ax+hy)h + (ax+hy)^2 b}{(hx+by)^3}\right]$$

State and Prove implicit function theorem for a real valued function of two real variables :→

Statement : Let $f(x,y)$ be a function of two variables x and y and let (a,b) be a point of its domain such that

(i) $f(a,b) = 0$

(ii) the partial derivatives f_x and f_y Exist and are Continuous in a Certain nhd of $f(a,b)$ and

(iii) $f_y(a,b) \neq 0$

then there Exists a rectangle $[a-h, a+h, b-k, b+k]$ about (a,b) such that for all $x \in [a-h, a+h]$, the Equation $f(x,y) = 0$ determines One and Only one Solution $y = g(x)$ lying in the interval $[b-k, b+k]$ with the following Properties

① $b = g(a)$

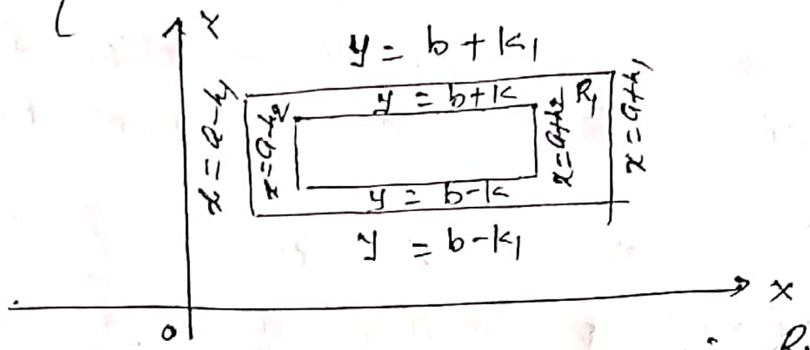
② $f[x, g(x)] = 0$ for all $x \in (a-h, a+h)$

③ $g(x)$ is derivable and both $g(x)$ and $g'(x)$ are Continuous in $(a-h, a+h)$

Proof :→ Let us suppose that we are given the Equation $f(x,y) = 0$ and try to Find the function $y = g(x)$ which identically Satisfies this equation in every region of values of x.

Let f_x; f_y be continuous in nbd

$R_1 = [a-h_1, a+h_1, b-k_1, b+k_1]$ of (a,b)



Since f_x, f_y are continuous in R_1, therefore f is continuous in R_1

Since f_y is continuous at (a,b) and $f_y(a,b) > 0$,

there exists a rectangle $R_2 = [a-h_2, a+h_2; b-k, b+k]$; $h_2 < h_1$, $k < k_1$ such that for every point (x,y) in this rectangle R_2, $f_y(x,y) > 0$

clearly $R_2 \subset R_1$

Now since $f_y(x,y) > 0$ for all $(x,y) \in R_2$

therefore for all $x \in [a-h_2, a+h_2]$ the function f of y strictly increases

from $b-k$ to $b+k$

Since $f(a,b) = 0$, therefore we have

$f(a, b-k) < 0$ —— (1)

$f(a, b+k) > 0$

Since f is continuous, these inequalities remain valid when we replace a by any number x which sufficiently close to a and since h_2 can be arbitrary small, we have the every justification to assume that the inequalities $f(x, b-k) < 0$

and $f(x, b+k) > 0$

are satisfied at every point x in the interval. (5)
$[a-h, a+h]$; $h < h_2$

Now, for every fixed value of x in $[a-h, a+h]$, the continuous function f of y strictly increases from a negative to a positive value of y increases from $b-k$ to $b+k$.

Hence a point y^* (say) can be found ~~between~~ between $b-k$ and $b+k$ for which $f(x, y^*) = 0$

thus there exists one and only one value of y for which the function vanishes.

therefore, for each value of x in $[a-h, a+h]$ there is uniquely determined value $y = g(x)$ for which $f(x,y) = 0$

this value of y is such that properties (1) and (2) are true.

this completes the proof of the existence and uniqueness of implicit function g.

(2) Continuity :- As we know g is continuous in $[a-h, a+h]$. for we have to prove

$$|g(x) - g(x_0)| < \epsilon \text{ when } |x - x_0| \leq \delta$$

where x_0 be any point $\in [a-h, a+h]$ and let

$$y_0 = f(x_0)$$

Let ϵ be any positive number, however small

Let $R' = [x_0-\delta_1, x_0+\delta_1; y_0-\epsilon, y_0+\epsilon]$ be a rectangle lying wholly within the rectangle

$R_1 = [a-h, a+h; b-k, b+k]$ as found above.

for this rectangle we can carry out the same as before in order to get a solution of

$$f(x,y) = 0$$

Since the solution was uniquely determined in R which encloses R', we see that $y = g(x)$ is also a solution in R'

Thus there exists an interval $[x_0-\delta, x_0+\delta]$, $\delta \le \delta_1$ such that for every value of x in this interval $g(x)$ lies between $y_0-\epsilon$ and $y_0+\epsilon$

Thus g is continuous at x_0 and hence in $[a-h, a+h]$

(3) ~~Differentiab~~ **Derivability** :→ Let x be a point in the interval $[a-h, a+h]$ and let $x+p$ $(p = \delta x)$ be another point of same interval.

Let $y = f(x)$ and

$y+q = f(x+p)$, $(q = \delta y)$

then $f(x,y) = 0$ and also $f(x+p, y+q) = 0$

On substraction $0 = f(x+p, y+q) - f(x,y)$

$$= \{f(x+p, y+q) - f(x+p, y)\} + \{f(x+p, y) - f(x,y)\}$$

Which by mean-value theorem becomes

$$= q f_y(x+p, y+\theta_1 q) + p f_x(x+\theta_2 p, y);$$

where $0 < \theta_1, \theta_2 < 1$

Since $f_y \ne 0$ in R and $(x+p, y+q)$ is a point of R, we have

$$\frac{q}{p} = -\frac{f_x(x+\theta_2 p, y)}{f_y(x+p, y+\theta_1 q)}$$

Since f_x and f_y are continuous,

hence $f_x(x+\theta_2 p, y) \longrightarrow f_x(x,y)$ and $f_y(x+p, y+\theta_1 q) \to f_y(x,y)$ as $(p,q) \to (0,0)$

Now $g'(x) = \lim_{p\to 0} \frac{g(x+p) - g(x)}{p} = \lim_{p\to 0} \frac{y+q-y}{p} = \lim_{p\to 0} \frac{q}{p}$

Hence from (2), in the limit when $(p,q) \to (0,0)$,

$$g'(x) = -\frac{f_x(x,y)}{f_y(x,y)}$$ thus g is

derivable and $g'(x) = -\frac{f_x(x,y)}{f_y(x,y)}$

Also the equation for $g'(x)$ shows that it is continuous. [this complete the proof]